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Coalescence of electromagnetic travelling waves in a saturated ferrite

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Abstract. We investigate how dissipation and nonlinearity affect an electromagnetic perturbation propagating into a saturated ferromagnet in the presence of an external magnetic field. We study the problem in $(1+1)$ and $(2+1)$ dimensions. It is found that at lowest order of the perturbation theory, the Burgers' equation in $(1+1)$ dimensions governs such dynamics. In $(2+1)$ dimensions we show that the phenomena obeys a nonlinear evolution equation (non-integrable) of Burgers type. We give exact solutions which describe in $(1+1)$ dimensions the propagation of a travelling electromagnetic wave and the coalescence of N travelling fronts and in $(2+1)$ dimensions the propagation of a nearly one-dimensional travelling front. We establish, in terms of the physical parameters of the system, whether breaking or diffusion of the initial perturbation dominates.

1. Introduction

The study of electromagnetic wave propagation in ferromagnet is not only interesting from a theoretical point of view but also from a practical point of view, particularly in connection with the behaviour of ferrite devices at microwave frequencies [1, 2] such as ferrite-loaded waveguides.

The propagation of electromagnetic waves in a ferromagnet obeys nonlinear equations of dispersion and dissipation. The linear theory has been investigated extensively in [3] and this approach provided a good explanation for phenomena such as cut-offs, resonances and wave-focusing.

Recently I Nakata began a rigorous study of the nonlinear case. In [4] he investigated propagation of nonlinear electromagnetic waves of long wavelength in a saturated ferromagnet taking into account nonlinearity and dispersion and in [5] examined the effect of dissipation on such propagation. The final result was the reduction of the evolution equations to a nonlinear integro-differential equation (non-integrable) of modified KdV type.

In this paper we investigate the effects of dissipation and nonlinearity on the propagation of a small electromagnetic perturbation in a saturated ferrite, in the presence of an external constant magnetic field, directly, and not only as a small correction to the evolution obtained considering nonlinearity and dispersion, which is the point of view taken in [5].

We do this using the reductive perturbation method. The stretching of the coordinates is based on a Gardner–Morikawa transformation characterized by one of the three phase velocities allowed by the linear system and different from that considered

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in [4]. For this phase velocity, the reductive perturbation method shows that breaking can be balanced only by dissipation. We study the problem in one and two spacial dimensions.

The main results obtained are: in $(1+1)$ dimensions (x and t variables) we prove that the dynamics of an electromagnetic perturbation of one initial static state of a saturated ferrite obeys the Burgers' equation. Hence an initial perturbation of the step profile type diffuses into a permanent shock Taylor-type profile and N initial perturbations coalesce. We characterize the velocity and amplitude of these travelling waves as functions of the physical parameters of the system. We show that if the angle between the direction of propagation of the perturbation and the external magnetic field is close to zero, diffusion is dominant. If this angle is close to $\pi/2$ the perturbation propagates without deformation. In $(2+1)$ dimensions (two space variables x and y and a time variable t), we prove that the evolution of a small electromagnetic perturbation obeys a nonlinear evolution equation of Burgers' type [6] (non-integrable), of which we give particular explicit solutions that describes a nearly one-dimensional travelling wave front. The transverse coordinate y is weaker than the x coordinate and we show that it must be orthogonal to the plane determined by the external (constant) magnetic field and the direction of propagation of the perturbation.

The paper is organized as follows. In section 2 we give the mathematical formulation of the system. In section 3 we study this system by using a perturbation theory in $(1+1)$ dimensions. In section 4 we construct the $(1+1)$ travelling waves solution and we show the coalescence of N of such waves. In section 5 we study the problem in $(2+1)$ dimensions. Finally in the appendix we give the technical details of the derivation of the evolution equation in the $(2+1)$ dimensional case.

2. Mathematical formulation of the system

The general form of Maxwell's equations in MKS units reads

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1}$$

$$\nabla \wedge \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{2.2}$$

in which $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ have their standard meaning. The constitutive equations in the ferromagnet for $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{H}$, $\mathbf{B}$ are given by

$$\mathbf{D} = \hat{\varepsilon}\mathbf{E} \tag{2.3}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{2.4}$$

where we shall assume that $\hat{\varepsilon}$ is the scalar permittivity of the ferromagnet, μ_0 is the magnetic permeability in vacuum, and $\mathbf{M}$ is the magnetization density in the ferromagnet. We considered a ferromagnet with saturated magnetization density. In the presence of an external magnetic field the magnetization density is governed by the torque equation [1]

$$\frac{\partial \mathbf{M}}{\partial t} = -\mu_0 \delta \mathbf{M} \wedge \mathbf{H} + \sigma \frac{[\mathbf{M} \wedge (\mathbf{M} \wedge \mathbf{H})]}{M}. \tag{2.5}$$

The second term on the right-hand side of equation (2.5) is a damping term which was first proposed by Landau and Lifchitz [7] for expressing the experimental fact that the magnetization $\mathbb{M}$ has a tendency to eventually line up with $\mathbb{H}$. In (2.5) $\sigma(\sigma<0)$ is a parameter which determines the magnitude of damping, δ is the gyromagnetic ratio and $M=|\mathbb{M}|$. Also in (2.5) we do not take into account either the term coming from the magnetic anisotropy or the one which represents the inhomogeneous exchange interaction. The first is neglected because we consider an isotropic ferromagnet and the second because the space scale associated with the electromagnetic perturbation considered here (typically that of electromagnetic waves in ferrites) substantially exceeds the space scale associated with the inhomogeneous exchange interaction (typically that of spin waves).

Taking the curl of (2.2) and using (2.1), (2.3) and (2.4) we have

$$-\nabla(\nabla\cdot\mathbb{H})+\nabla^2\mathbb{H}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbb{H}+\mathbb{M}) \tag{2.6}$$

where $c=(\hat{\varepsilon}\mu_0)^{-1/2}$ is the speed of light based on the dielectric constant of the ferromagnet. If the magnetization were zero, then $\nabla\cdot\mathbb{H}=0$ and (2.6) would be the linear wave equation, satisfied by isotropic, dispersionless transverse waves, propagating at speed c. Such is not the case and equations (2.5) and (2.6) are a system of complicated nonlinear partial differential equations for $\mathbb{M}$ and $\mathbb{H}$ describing electromagnetic wave propagation in a saturated ferromagnet, which we are going to study with a perturbation theory.

Finally, we observe that we can obtain the dispersion relation of the system. Let us consider in the $(1+1)$-dimensional case (x, t) a constant solution of (2.5), (2.6) given by the constant vectors $\boldsymbol{H}_0/\mu_0\delta$, $\boldsymbol{M}_0/\mu_0\delta$ (the factor $1/\mu_0\delta$ is included for convenience) with

$$\boldsymbol{H}_0=\alpha\boldsymbol{M}_0 \tag{2.7}$$

and with $\boldsymbol{M}_0$, $\boldsymbol{H}_0$ in the form $\boldsymbol{M}_0=M_0(\cos\varphi, \sin\varphi, 0)$ and $\boldsymbol{H}_0=H_0(\cos\varphi, \sin\varphi, 0)$.

We suppose another solution of (2.5), (2.6) as being a small perturbation of (2.7) given by

$$\mathbb{M}=\boldsymbol{M}_0+\boldsymbol{m}\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)ct)} \tag{2.8}$$

$$\mathbb{H}=\boldsymbol{H}_0+\boldsymbol{h}\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)ct)} \tag{2.9}$$

where $\boldsymbol{m}$, $\boldsymbol{h}$ are real vectors of components (m_x, m_y, m_z) and (h_x, h_y, h_z) and k, and ω are, respectively, the wavenumber and the frequency of the wave.

Substituting (2.8) and (2.9) in (2.5) and (2.6) and disregarding nonlinear terms in $\boldsymbol{m}$, $\boldsymbol{h}$ we have the system

$$\mathrm{i}\omega c\boldsymbol{m}=\boldsymbol{M}_0\wedge(\boldsymbol{h}-\alpha\boldsymbol{m})+\frac{\sigma}{\mu_0\delta M_0}\boldsymbol{M}_0\wedge(\boldsymbol{M}_0\wedge(\boldsymbol{h}-\alpha\boldsymbol{m})) \tag{2.10}$$

$$\omega^2(h_x+m_x)=0 \tag{2.11}$$

$$\omega^2(h_y+m_y)=k^2h_y \tag{2.12}$$

$$\omega^2(h_z+m_z)=k^2h_z. \tag{2.13}$$

From (2.11), (2.12) and (2.13) we have m_x, m_y, m_z as functions of h_x, h_y, h_z, and their substitution in (2.10) give us a linear system of equations for the unknowns

h_x, h_y, h_z, which reads

$$h_x\{-\omega^2+p_y r_y(\omega^2+\alpha k^2)\}-h_y\{p_y r_x[\omega^2(1+\alpha)-\alpha k^2]\} + h_z\{p_y[\omega^2(1+\alpha)-\alpha k^2]\}=0 \tag{2.14}$$

$$-h_x\{p_x r_y(\omega^2+\alpha k^2)\}+h_y\{k^2-\omega^2+p_x r_x[\omega^2(1+\alpha)-\alpha k^2]\} + h_z\{p_x[\omega^2(1+\alpha)-\alpha k^2]\}=0 \tag{2.15}$$

$$h_x\{p_y(\omega^2+\alpha k^2)\}-h_y\{p_x[\omega^2(1+\alpha)-\alpha k^2]\} + h_z\{k^2-\omega^2+(p_x r_x+p_y r_y)[\omega^2(1+\alpha)-\alpha k^2]\}=0 \tag{2.16}$$

where the auxiliary vectors $\boldsymbol{p}$ and $\boldsymbol{r}$ are defined by

$$\boldsymbol{p}=-\frac{\mathrm{i}}{\omega c}\boldsymbol{M}_0 \qquad \boldsymbol{r}=\sigma\boldsymbol{M}_0.$$

This system of equations for h_x, h_y and h_z has a non-trivial solution if and only if its determinant is zero. This conditions yields, for the dispersion relation of the system (2.5), (2.6) in $(1+1)$

$$\omega^2c^2(\omega^2-k^2)^2-M_0^2(1+\gamma^2M_0^2)[\omega^2(1+\alpha)-\alpha k^2][\omega^2(1+\alpha)-k^2(\alpha+\sin^2\varphi)] + \mathrm{i}\sigma\omega c M_0^2(\omega^2-k^2)[2\omega^2(1+\alpha)-k^2(2\alpha+\sin^2\varphi)]=0. \tag{2.17}$$

Now, we search for solutions of (2.17) such that the phase velocity $\omega(k)/k$ remains finite in the long wavelength limit $k=0(\varepsilon)$ with $\varepsilon\ll 1$:

$$\frac{\omega(k)}{k}=\omega_0+k\omega_1+\ldots.$$

There are two solutions of this type for (2.17), which read

$$\frac{\omega^2(k)}{k^2}=\frac{\alpha}{1+\alpha}+0(\varepsilon) \tag{2.18}$$

$$\frac{\omega^2(k)}{k^2}=\frac{\alpha+\sin^2\varphi}{1+\alpha}+0(\varepsilon). \tag{2.19}$$

In the perturbation theory developed in section 3 we find, naturally, the velocity (2.19) and our results are valid for perturbations having a long wavelength limit of phase velocity given by (2.19).

In the $(2+1)$-dimensional case an identical procedure (but, more laborious algebraically) yields, at order ε^0, the relations

$$\frac{\omega^2(k,l)}{k^2+l^2}=\left(\frac{\alpha}{1+\alpha}\right)^{1/2}+0(\varepsilon) \tag{2.20}$$

$$\frac{\omega^2(k,l)}{k^2+l^2}=\frac{1}{1+\alpha}\left(\alpha+(s^x)^2+\frac{(s^x l-s^y k)^2}{k^2+l^2}\right)+0(\varepsilon) \tag{2.21}$$

where $s=\boldsymbol{M}_0/M_0$, and k, l of order ε are the wavenumbers in the x, y directions. Assuming $l\ll k$, we obtain from (2.21) with $\sin^2\phi=(s^x)^2+(s^y)^2$

$$\frac{\omega^2}{k^2}=\frac{\alpha+\sin^2\phi}{1+\alpha}+0(\varepsilon)$$

which is the velocity found in section 5, formula (5.9) with $\phi=\varphi+\pi$.

3. Perturbation scheme and the (1 + 1) Burger equation

In this section we study the system (2.5), (2.6) by using a perturbation theory in which the solution is expanded as a formal asymptotic series. For details of this type of approximation see [8].

Let us consider that there exists a solution of (2.5), (2.6) expanded in a series in powers of the small parameter ε.

$$\mathbb{M}(\xi, \tau) = \sum_{n=0}^{\infty} \varepsilon^n \boldsymbol{M}_n(\xi, \tau) \tag{3.1}$$

$$\mathbb{H}(\xi, \tau) = \sum_{n=0}^{\infty} \varepsilon^n \boldsymbol{H}_n(\xi, \tau). \tag{3.2}$$

The fields $\boldsymbol{M}_0$ and $\boldsymbol{H}_0$ characterize the initial state of the system and the small parameter ε measures the normalized amplitude of the applied magnetic perturbation. In (3.1), (3.2) $\boldsymbol{M}_n$, $\boldsymbol{H}_n$ are three-component vectors, $\boldsymbol{M}_n = (M_n^x, M_n^y, M_n^z)$, $\boldsymbol{H}_n = (H_n^x, H_n^y, H_n^z)$. This solution is considered as a function of the slow variables ξ, τ introduced through the stretching

$$\xi = \varepsilon(x - \beta t) \tag{3.3}$$

$$\tau = \varepsilon^2 t \tag{3.4}$$

where the velocity β will be determined later as a solvability condition of equations (2.5), (2.6). Substituting (3.1), (3.2), (3.3) and (3.4) (rescaling $\mathbb{M}$, $\mathbb{H}$, t, σ into $(\mu_0\delta/c)\mathbb{M}$, $(\mu_0\delta/c)\mathbb{H}$, ct and $\sigma/\mu_0\delta$) into (2.5), (2.6) and collecting powers of ε^n we can solve it order by order. The conditions on $\boldsymbol{M}_n$, $\boldsymbol{H}_n$ for $\xi \to -\infty$ are: $\boldsymbol{M}_n$, $\boldsymbol{H}_n$ and all their derivatives go to zero for $n = 2, 3, 4, \ldots$ $\boldsymbol{M}_0$, $\boldsymbol{H}_0$ go to $\boldsymbol{m}$, $\boldsymbol{h}$ and $\boldsymbol{M}_1 \to 0$, $\boldsymbol{H}_1 \to \boldsymbol{l}$ where $\boldsymbol{l}$ is a constant vector parallel to $\boldsymbol{m}$. A more general choice for the limits of $\boldsymbol{M}_1$, $\boldsymbol{H}_1$ at $\xi \to -\infty$ would complicate drastically the calculus without giving more relevant conclusions.

At order zero we have the system:

$$\boldsymbol{M}_0 \wedge \boldsymbol{H}_0 - \sigma\left[\frac{\boldsymbol{M}_0 \cdot \boldsymbol{H}_0}{M_0} \boldsymbol{M}_0 - M_0 \boldsymbol{H}_0\right] = 0 \tag{3.5}$$

$$\frac{\partial^2}{\partial \xi^2}(H_0^x + M_0^x) = 0 \tag{3.6}$$

$$\frac{\partial^2}{\partial \xi^2}(\gamma H_0^y + M^y) = 0 \tag{3.7}$$

$$\frac{\partial^2}{\partial \xi^2}(\gamma H_0^z + M_0^z) = 0 \tag{3.8}$$

where $\gamma = 1 - \beta^{-2}$. Dot denotes, as usual, the scalar product and $M_0 = |\boldsymbol{M}_0|$, $H_0 = |\boldsymbol{H}_0|$.

Multiplying (dot product) equation (3.5) by $\boldsymbol{H}_0$ we obtain

$$\sigma\left[\frac{(\boldsymbol{M}_0\cdot\boldsymbol{H}_0)^2}{M_0}-M_0H_0^2\right]=0 \tag{3.9}$$

$$(\boldsymbol{M}_0\cdot\boldsymbol{H}_0)^2=M_0^2H_0^2 \tag{3.10}$$

and this implies that $\boldsymbol{M}_0$ and $\boldsymbol{H}_0$ are colinear, which can be written as

$$\boldsymbol{H}_0=\lambda(\xi,\tau)\boldsymbol{M}_0. \tag{3.11}$$

By integration of equations (3.6), (3.7) and (3.8) we obtain the relations (using (3.11))

$$M_0^x=\frac{1+\alpha}{1+\lambda}m^x \tag{3.12}$$

$$M_0^y=\frac{1+\alpha\gamma}{1+\alpha\lambda}m^y \tag{3.13}$$

$$M_0^z=0 \tag{3.14}$$

where

$$\alpha=\lim_{\xi\to-\infty}\lambda$$

and we suppose that $\mu=1+\alpha\gamma\neq 0$. We have appropriately chosen the Cartesian coordinate system such that $\boldsymbol{m}$ can be written as a vector of the form $\boldsymbol{m}=(m^x,m^y,0)$.

At order ε the equation (2.5) gives (using the results of order zero)

$$-\beta\frac{\partial}{\partial\xi}\boldsymbol{M}_0=-\boldsymbol{M}_0\wedge(\boldsymbol{H}_1-\lambda\boldsymbol{M}_1)+\frac{\sigma}{M_0}\boldsymbol{M}_0\wedge[\boldsymbol{M}_0\wedge(\boldsymbol{H}_1-\lambda\boldsymbol{M}_1)]. \tag{3.15}$$

Multiplying by $\boldsymbol{M}_0$ we obtain that M_0 is a constant vector, thus M_0^2 is a constant and $M_0^2=m^2$. Using (3.12), (3.13) and (3.14) we obtain the equation

$$\frac{(1+\alpha)^2}{(1+\lambda)^2}(m^x)^2+\frac{(1+\alpha\gamma)^2}{(1+\gamma\lambda)^2}(m^y)^2=(m^x)^2+(m^y)^2. \tag{3.16}$$

Consequently λ satisfies an equation of second order with constant coefficients, and thus it is a constant and $\lambda=\alpha$. Therefore $\boldsymbol{M}_0=\boldsymbol{m}$ and $\boldsymbol{H}_0=\alpha\boldsymbol{m}$ are constant vectors characterizing the initial static state of the system. Using this fact in (3.15) we obtain, after multiplication by $(\boldsymbol{H}_1-\alpha\boldsymbol{M}_1)$, that $(\boldsymbol{H}_1-\alpha\boldsymbol{M}_1)$ and $\boldsymbol{m}$ are colinear:

$$[\boldsymbol{m}\cdot(\boldsymbol{H}_1-\alpha\boldsymbol{M}_1)]^2=(\boldsymbol{m})^2(\boldsymbol{H}_1-\alpha\boldsymbol{M}_1)^2. \tag{3.17}$$

Let us thus define $f(\xi,\tau)$ as (the factor $(1+\alpha)\mu$ is introduced for convenience)

$$\boldsymbol{H}_1-\alpha\boldsymbol{M}_1=(1+\alpha)\mu f(\xi,\tau)\boldsymbol{m}. \tag{3.18}$$

At the same order ε we obtain from (2.6):

$$\frac{\partial^2}{\partial\xi^2}(H_1^x+M_1^x)-\frac{2}{\beta}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\tau}(H_0^x+M_0^x)=0 \tag{3.19}$$

$$\frac{\partial}{\partial\xi^2}(\gamma H_1^y+M_1^y)-\frac{2}{\beta}\frac{\partial}{\partial\xi}\frac{\partial}{\partial\tau}(H_0^y+M_0^y)=0 \tag{3.20}$$

$$\frac{\partial}{\partial \xi^2}(\gamma H_1^z + M_1^z) - \frac{2}{\beta}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau}(H_0^z + M_0^z) = 0. \tag{3.21}$$

Using the fact that $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$ are constant and that for $\xi \to -\infty$ $\boldsymbol{M}_1 \to \boldsymbol{0}$ and $\boldsymbol{H}_1 \to \boldsymbol{l}$ ($\boldsymbol{l}$ parallel to $\boldsymbol{m}$) and also that all derivatives of $\boldsymbol{M}_n, \boldsymbol{H}_n (\forall n)$ go to zero we can integrate equations (3.19), (3.20) and (3.21)

$$H_1^x + M_1^x = l^x \tag{3.22}$$

$$\gamma H_1^y + M_1^y = \gamma l^y \tag{3.23}$$

$$\gamma H_1^z + M_1^z = \gamma l^z. \tag{3.24}$$

The limit $\xi \to -\infty$ in (3.18) gives

$$l^x = (1+\alpha)\mu f_0 m^x \tag{3.25}$$

$$l^y = (1+\alpha)\mu f_0 m^y \tag{3.26}$$

$$l^z = 0 \tag{3.27}$$

or

$$f_0 = \frac{l}{m\mu(1+\alpha)} \qquad l = |\boldsymbol{l}| \tag{3.28}$$

where f_0 is $f(\xi, \tau)$ for $\xi \to -\infty$. Using (3.25), (3.26), (3.27) and (3.18) in (3.22), (3.23), (3.24) we obtain finally

$$H_1^x = \mu m^x (f + \alpha f_0) \qquad M_1^x = \mu m^x (f_0 - f) \tag{3.29}$$

$$H_1^y = (1+\alpha) m^y (f + \alpha\gamma f_0) \qquad M_1^y = \gamma(1+\alpha) m^y (f_0 - f) \tag{3.30}$$

$$H_1^z = 0 \qquad M_1^z = 0. \tag{3.31}$$

The above equations constitute the complete solution in the order ε. In the next order the equations (2.5), (2.6) give

$$-\beta \frac{\partial}{\partial \xi}\boldsymbol{M}_1 = -\boldsymbol{m} \wedge (\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2) - \boldsymbol{M}_1 \wedge \boldsymbol{H}_1 + \sigma \frac{\boldsymbol{m}}{m} \wedge [\boldsymbol{m} \wedge (\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2) + \boldsymbol{M}_1 \wedge \boldsymbol{H}_1] \tag{3.32}$$

$$M_2^x = -H_2^x \tag{3.33}$$

$$M_2^y = -\gamma H_2^y + \frac{2(1+\alpha)}{\beta} m^y \int_{-\infty}^{\xi} \frac{\partial f}{\partial \tau}(\xi', \tau)\, \mathrm{d}\xi' \tag{3.34}$$

$$M_2^z = -\gamma H_2^z. \tag{3.35}$$

Introducing now the parameter φ through

$$m^x = m \cos \varphi \tag{3.36}$$

$$m^y = m \sin \varphi \tag{3.37}$$

we have from (3.32), (3.33), (3.34) and (3.35) the following linear system for the $\boldsymbol{H}_2$ components

$$\beta\mu m(\cos\varphi)\frac{\partial f}{\partial\xi} = -\mu m(\sin\varphi)H_2^z + \sigma\sin\varphi[\mu m(\cos\varphi)H_2^y - (1+\alpha)m(\sin\varphi)H_2^x - \Phi] \quad (3.38)$$

$$\beta\gamma(1+\alpha)m(\sin\varphi)\frac{\partial f}{\partial\xi} = \mu m(\cos\varphi)H_2^z + \sigma(\cos\varphi)[-\mu m(\cos\varphi)H_2^y + (1+\alpha)m(\sin\varphi)H_2^x + \Phi] \quad (3.39)$$

$$0 = -\mu(\cos\varphi)H_2^y + (1+\alpha)(\sin\varphi)H_2^x - \sigma\mu H_2^z + \frac{\Phi}{m} \quad (3.40)$$

where Φ is defined by

$$\Phi(\xi,\tau) = (1+\alpha)m^2(\cos\varphi\sin\varphi)\left\{\frac{2\alpha}{\beta^3}\int_{-\infty}^{\xi}\frac{\partial f}{\partial\tau}(\xi',\tau)\,\mathrm{d}\xi' + (1-\gamma)\mu f(f-f_0)\right\}. \quad (3.41)$$

From linear algebra we can see that the system (3.38), (3.39), (3.40) has a non-trivial solution if and only if, the velocity β is given by

$$\beta = \left(\frac{\alpha+\sin^2\varphi}{\alpha+1}\right)^{1/2}. \quad (3.42)$$

Under this condition we obtain that the non-trivial solution satisfies

$$H_2^z = \frac{-(\sin\varphi)(\cos\varphi)}{\beta\mu(1+\sigma^2)}\frac{\partial f}{\partial\xi} \quad (3.43)$$

$$\mu(\cos\varphi)H_2^y - (1+\alpha)(\sin\varphi)H_2^x = \frac{\Phi}{m} + \sigma\frac{(\sin\varphi)(\cos\varphi)}{\beta(1+\sigma^2)}\frac{\partial f}{\partial\xi}. \quad (3.44)$$

Now multiplying equation (2.5) at order ε^2 by $\boldsymbol{m}$ we obtain

$$-\beta\frac{\partial}{\partial\xi}\boldsymbol{m}\cdot\boldsymbol{M}_2 + \frac{\partial}{\partial\tau}\boldsymbol{m}\cdot\boldsymbol{M}_1 = -\boldsymbol{m}\cdot(\boldsymbol{M}_1\wedge\boldsymbol{H}_2 + \boldsymbol{M}_2\wedge\boldsymbol{H}_1) + \frac{\sigma}{m}\boldsymbol{m}\cdot[\boldsymbol{M}_1\wedge\boldsymbol{U}] \quad (3.45)$$

with $\boldsymbol{U} = \boldsymbol{m}\wedge(\boldsymbol{H}_2 - \alpha\boldsymbol{M}_2) + \boldsymbol{M}_1\wedge\boldsymbol{H}_1$.

Using in (3.45) the results from previous orders we finally obtain a nonlinear evolution equation for the function $f(\xi,\tau)$

$$\frac{\partial f}{\partial\tau} + \frac{\sigma}{1+\sigma^2}\frac{\cos^2\varphi}{2m(1+\alpha)^2}\frac{\partial^2 f}{\partial\xi^2} + \frac{(1+\alpha)^{1/2}\sin^2\varphi\cos^2\varphi}{(\alpha+\sin^2\varphi)^{3/2}}\left[\frac{3}{2}f\frac{\partial f}{\partial\xi} - f_0\frac{\partial f}{\partial\xi}\right] = 0. \quad (3.46)$$

This is the well known Burgers' equation [9]. This equation includes nonlinearity and dissipation in the simplest way, and it can be thought of as a nonlinear version of the heat equation.

4. Travelling-wave solutions and their coalescence

Our aim in this section is to calculate solutions of the Burgers' equation (3.46). Before doing this we will write it in its canonical form. First of all we return to the original fields, making the inverse transformations

$$\tau \to c\tau \qquad f \to \frac{\mu_0 \delta}{c} f \qquad \text{and} \qquad \sigma \to \frac{\sigma}{\mu_0 \delta}$$

$$p\,\frac{\partial f}{\partial \tau} - q\,\frac{\partial^2 f}{\partial \xi^2} - rf\,\frac{\partial f}{\partial \xi} + s\,\frac{\partial f}{\partial \xi} = 0 \tag{4.1}$$

with p, q, r and s given by

$$p = \frac{1}{\mu_0 \delta} \tag{4.2}$$

$$q = \frac{-\sigma c \cos^2 \varphi}{(\mu_0^2 \delta^2 + \sigma^2) 2m(1+\alpha)^2} \tag{4.3}$$

$$r = -\frac{3}{2}\,\frac{(\sin^2 \varphi)(\cos^2 \varphi)(1+\alpha)^{1/2}}{(\alpha + \sin^2 \varphi)^{3/2}} \tag{4.4}$$

$$s = -\frac{(\sin^2 \varphi)(\cos^2 \varphi)(1+\alpha)^{1/2}}{(\alpha + \sin^2 \varphi)^{3/2}}\, f_0 . \tag{4.5}$$

Note that q is a real positive constant ($\sigma < 0$), which goes to zero for $\sigma \to 0$ ($\varphi \neq \pi$) thus in this case breaking is dominant. In the case $\sigma \neq 0$, $\varphi \sim 0$ or $\varphi \sim \pi$ we would expect diffusion to dominate. For $\varphi \sim \pi/2$ all the coefficients are zero and the perturbation propagates without deformation.

A Galilean transformation allows us to eliminate the last term of (3.46) and to simplify also the rest of the coefficients. It reads ($\varphi \neq \pm\pi$, $\pm\pi/2$)

$$X = \alpha_0 \xi + \alpha_1,\ \tau \tag{4.6}$$

$$T = \alpha_2 \tau \tag{4.7}$$

with $\alpha_0 = -r^{-1}$, $\alpha_1 = s(rp)^{-1}$, $\alpha_2 = p^{-1}$. We obtain thus

$$\frac{\partial f}{\partial T} + f\,\frac{\partial f}{\partial X} - \rho\,\frac{\partial^2 f}{\partial X^2} = 0 \tag{4.8}$$

where $f \to f_0$ for $X \to -\infty$ and the coefficient of diffusion ρ is given by

$$\rho = \frac{-2\sigma c}{9m \sin^4 \varphi \cos^2 \varphi (\mu_0^2 \delta^2 + \sigma^2)} \left(\frac{\alpha + \sin^2 \varphi}{\alpha + 1} \right)^3 . \tag{4.9}$$

From the bilinear form of equation (4.8) [10, 11] we can find its one travelling wave solution which in laboratory coordinates reads

$$f(x, t) = f_0\,\frac{e^\eta}{1+e^\eta} = \frac{f_0}{2}\left(1 + \tanh\left(\frac{\eta}{2}\right)\right) \tag{4.10}$$

with

$$\eta = A(x - vt).$$

$$A = \frac{3}{2}\frac{h}{c}\frac{\mu_0^2\delta^2 + \sigma^2}{\sigma}\left(\frac{\alpha+1}{\alpha+\sin^2\varphi}\right)^{1/2} \tag{4.11}$$

$$v = c\left\{\left(\frac{\alpha+\sin^2\varphi}{\alpha+1}\right)^{1/2} - \frac{\mu_0\delta h\cos^2\varphi}{4m(\alpha+1)^{3/2}(\alpha+\sin^2\varphi)^{1/2}}\right\} \tag{4.12}$$

where h is a measure of the magnetic perturbation for $x \to -\infty$

$$h = \varepsilon l = \varepsilon f_0 m\mu(1+\alpha). \tag{4.13}$$

The function f is the Taylor shock profile solution of Burgers' equation which is generally named a 'shock profile'. Note that f is completely characterized as a function of the initial input f_0 and the physical parameters of the problem.

Another form of solution to (4.8) is found using the nonlinear Hopf–Cole transformation which reduces (4.8) to a linear heat equation [12]. In our case the Hopf–Cole transformation and the associated heat equation reads

$$f(X, T) = -2\rho\frac{\partial}{\partial X}\ln\varphi(X, T) \tag{4.14}$$

$$\frac{\partial\varphi}{\partial T} = \rho\frac{\partial^2\varphi}{\partial X^2}. \tag{4.15}$$

Then, instead of the nonlinear equation (4.8) we can study the linear equation (4.15). For example, the solution of the initial value problem of (4.8) is reduced to solving the following three steps: first, knowing $f(X, 0) = f(X)$ we evaluate $\varphi(X, 0)$ from (4.14). Second, via (4.15) and using the Fourier transform method we evaluate $\varphi(X, T)$. Finally, we recover $f(X, T)$ from (4.14). The final result is in our case

$$f(X, T) = \frac{\displaystyle\int_{-\infty}^{+\infty}\left(\frac{X - X'}{T}\right)\exp\left(-\frac{S}{2\rho}\right)\mathrm{d}X'}{\displaystyle\int_{-\infty}^{+\infty}\exp\left(-\frac{S}{2\rho}\right)\mathrm{d}X'} \tag{4.16}$$

$$S(X'; X, T) = \int_0^{X'} f(l)\,\mathrm{d}l + \frac{(X - X')^2}{2T}. \tag{4.17}$$

Many types of solution can be obtained from (4.16) choosing appropriately $f(X)$ in (4.17). For $f(X)$ a step function:

$$f(X) = \begin{cases} 0 & X > 0 \\ f_0 & X < 0 \qquad (f_0 > 0) \end{cases} \tag{4.18}$$

we obtain

$$f(X, T) = \frac{f_0}{2}\left\{1 - \tanh\left[\frac{f_0}{4\rho}\left(X - \frac{f_0 T}{2}\right) + \theta(X)\right]\right\} \tag{4.19}$$

with $\theta(X)$ given by

$$\exp\theta(X)=\frac{\int_{-X/\sqrt{2\rho T}}^{\infty}\exp(-\xi^2)\,\mathrm{d}\xi}{\int_{(X-f_0T)/\sqrt{2\rho T}}^{\infty}\exp(-\xi^2)\,\mathrm{d}\xi}. \tag{4.20}$$

The interest of this solution, corresponding to an initial step, is that it diffuses into a steady profile for $T\to\infty$ ('far-field limit'). In fact, we can see from (4.20) that for X/T fixed between the limits

$$0<\frac{X}{T}<f_0 \tag{4.21}$$

the function $\theta(X)\to 0$ as $T\to\infty$ and we obtain

$$f(X,T)=\frac{f_0}{2}\left\{1-\tanh\left[\frac{f_0}{4\rho}\left(X-\frac{f_0T}{2}\right)\right]\right\}. \tag{4.22}$$

This asymptoptic form of the solution obtained via the Hopf–Cole transformations is exactly the expression (4.10) if we write it in laboratory coordinates x, t.

The expression (4.10) for f allows us to obtain expressions for all x and t for the full magnetization perturbation and the full magnetic perturbation of order ε defined by

$$\boldsymbol{H}(x,t)=\varepsilon\boldsymbol{H}_1(x,t) \tag{4.23}$$

$$\boldsymbol{M}(x,t)=\varepsilon\boldsymbol{M}_1(x,t) \tag{4.24}$$

$$\boldsymbol{H}(x,t)=\frac{h\cot\varphi}{2(1+\alpha)}\left\{\sin\varphi\left(1+2\alpha+\tanh\frac{\eta}{2}\right),\right.$$

$$\left.\cos\varphi\left((1+2\alpha)\sin^2\varphi-(\alpha+\sin^2\varphi)\tanh\frac{\eta}{2}\right),0\right\} \tag{4.25}$$

$$\boldsymbol{M}(x,t)=\frac{h\cot\varphi}{1+\alpha}\,\frac{1}{1+\mathrm{e}^{\eta}}(\sin\varphi,-\cos\varphi,0). \tag{4.26}$$

The N travelling wave solutions corresponding to N initial inputs $f_0>f_1>\ldots>f_N$ for $x\to-\infty$ are given by

$$f=\frac{\sum_{i=1}^{N}f_i\,\mathrm{e}^{\eta_i}}{1+\sum_{i=1}^{N}f_i\,\mathrm{e}^{\eta_i}} \tag{4.27}$$

with

$$\eta_i=-ah_i(x-v_it).$$

The expressions for h_i, a and v_i are

$$h_i=\varepsilon f_i\mu m(1+\alpha) \tag{4.28}$$

$$a=\frac{3}{2c}\,\frac{\mu_0^2\delta^2+\sigma^2}{-\sigma}\left(\frac{\alpha+1}{\alpha+\sin^2\varphi}\right)^{1/2} \tag{4.29}$$

$$v_i=c\left(\frac{\alpha+\sin^2\varphi}{\alpha+1}\right)^{1/2}-\frac{\mu_0\delta_c\cos^2\varphi}{m(\alpha+1)^{3/2}(\alpha+\sin^2\varphi)^{1/2}}\left(h_0-\frac{3}{4}h_i\right) \tag{4.30}$$

and $\boldsymbol{H}(x,t)$, $\boldsymbol{M}(x,t)$ have the following forms

$$H^x(x,t)=\frac{\cos\varphi}{1+\alpha}\left\{\frac{\sum_{i=1}^N h_i\,\mathrm{e}^{-ah_i(x-v_it)}}{1+\sum_{i=1}^N \mathrm{e}^{-ah_i(x-v_it)}}+\alpha h_0\right\} \tag{4.31}$$

$$H^y(x,t)=\frac{1}{(\alpha+1)\sin\varphi}\left\{(\alpha+\sin^2\varphi)\frac{\sum_{i=1}^N h_i\,\mathrm{e}^{-ah_i(x-v_it)}}{1+\sum_{i=1}^N \mathrm{e}^{-ah_i(x-v_it)}}-\alpha h_0\cos^2\varphi\right\} \tag{4.32}$$

$$H^z(x,t)=M^z(x,t)=0 \tag{4.33}$$

$$M^x(x,t)=\frac{\cos\varphi\sin\varphi}{(\alpha+1)\sin\varphi}\left\{h_0-\frac{\sum_{i=1}^N h_i\,\mathrm{e}^{-ah_i(x-v_it)}}{1+\sum_{i=1}^N \mathrm{e}^{-ah_i(x-v_it)}}\right\} \tag{4.34}$$

$$M^y(x,t)=-\frac{\cos^2\varphi}{(\alpha+1)\sin\varphi}\left\{h_0-\frac{\sum_{i=1}^N h_i\,\mathrm{e}^{-ah_i(x-v_it)}}{1+\sum_{i=1}^N \mathrm{e}^{-ah_i(x-v_it)}}\right\}. \tag{4.35}$$

It is easy to see because $v_0>v_1>\ldots>v_N$ that the expressions (4.31) to (4.35) represent the coalescence of N travelling fronts. We note that it is natural that the parameter ε appears in the formulas for $\boldsymbol{H}$ and $\boldsymbol{M}$, since these expressions are valid in laboratory coordinates x and t. For certain systems the smallness of ε is made precise relating it with some (small) physical parameter of the model (for example, with the wavenumber when we consider long wave in shallow water). Here such identification is not possible and we identify it with the size of the initial perturbation.

5. Extension to x, y, t: a multidimensional nonlinear diffusion equation

In the $(1+1)$-dimensional case the x (or ξ) axis has been chosen along the direction of propagation of the initial perturbation, the very existence of which introduces a geometrical anisotropy in the medium. This fact, can be studied by considering the system in $(2+1)$ dimensions: two space variables x and y and a time variable t where the transverse coordinate y is weaker than the x coordinate, and represents a kind of transversal perturbation.

Let us consider that there exists a solution of (2.5), (2.6) expanded in a series in powers of ε given by

$$\mathbf{M}(\xi,\zeta,\tau)=\boldsymbol{M}_0(\xi,\zeta,\tau)+\sum_{n=1}^{\infty}\varepsilon^{n+1}\boldsymbol{M}_n(\xi,\zeta,\tau)=\boldsymbol{M}_0+\varepsilon^2\boldsymbol{M}_1+\varepsilon^3\boldsymbol{M}_2+\ldots \tag{5.1}$$

$$\mathbf{H}(\xi,\zeta,\tau)=\boldsymbol{H}_0(\xi,\zeta,\tau)+\sum_{n=1}^{\infty}\varepsilon^{n+1}\boldsymbol{H}_n(\xi,\zeta,\tau)=\boldsymbol{H}_0+\varepsilon^2\boldsymbol{H}_1+\varepsilon^3\boldsymbol{H}_2+\ldots. \tag{5.2}$$

This solution is considered as a function of slow variables ξ, ζ and τ introduced through the stretching

$$\xi=\varepsilon^2(x-vt) \tag{5.3}$$

$$\zeta=\varepsilon^3 y \tag{5.4}$$

$$\tau=\varepsilon^4 t. \tag{5.5}$$

A crucial point is the expression for ζ because in it lies the definition of the weak dependence of the fields parameters on the coordinate y. In the analogous problem studied in water theory [13] the weak coordinate y is in the plane determined by the water at rest (this is the initial static state for this type of system) and it is orthogonal to the direction of propagation of the carrier wave. Here, we do not know the relative orientation of the transverse y (or ζ) coordinate in the three-dimensional space x, y, z where the ferrite is immersed. This relative orientation and also the velocity v in (5.3) will be determined as solvability conditions of the system.

Rescaling $\mathbf{M}$, $\mathbf{H}$, t and σ such as in the $(1+1)$-dimensional case and solving (2.5) and (2.6) order by order in ε we obtain the following principal results (the precise assumptions made on the limits of M_n and H_n at infinity and the technical steps involved are given in the Appendix):

(i) $\boldsymbol{M}_0$ and $\boldsymbol{H}_0$ are constant vectors characterizing the initial static state

$$\boldsymbol{M}_0=\boldsymbol{m} \tag{5.6}$$

$$\boldsymbol{H}_0=\alpha\boldsymbol{m} \tag{5.7}$$

where $\boldsymbol{m}=(m^x, m^y, m^z)$. The vectors $\boldsymbol{H}_1$ and $\boldsymbol{M}_1$ are related by

$$\boldsymbol{H}_1-\alpha\boldsymbol{M}_1=(1+\alpha)\mu f(\xi, \zeta, \tau)\boldsymbol{m} \tag{5.8}$$

where $f(\xi, \zeta, \tau)$ is an arbitrary function of ξ, ζ, τ.

(ii) One first compatibility condition gives the velocity v as

$$v^2=\frac{\alpha+\sin^2\varphi}{1+\alpha} \tag{5.9}$$

where the angle φ verifies

$$\begin{aligned} &m^x=m\sin\varphi \\ &(m^y)^2+(m^z)^2=m^2\cos^2\varphi \qquad m=|\boldsymbol{m}|. \end{aligned} \tag{5.10}$$

(iii) One second compatibility condition gives

$$m^y=0. \tag{5.11}$$

This equation shows that the y (or ζ) coordinate is orthogonal to the plane determined by the external magnetic field $\boldsymbol{H}_0$ and the direction of propagation of the perturbation: x (or ξ).

(iv) Finally we obtain as evolution equation for $f(\xi, \zeta, \tau)$ the $(2+1)$-dimensional Burger equation, also named in gas theory [14] the Zabolotskaya–Khokhlov equation:

$$\left(Af_\tau+B\left(f-\frac{2}{3}f_0\right)f_\xi+Cf_{\xi\xi}\right)_\xi=Df_{\zeta\zeta} \tag{5.12}$$

with

$$A=1 \tag{5.13}$$

$$B=\frac{3}{2}(\sin^2\varphi)(\cos^2\varphi)\frac{(\alpha+1)^{1/2}}{(\alpha+\sin^2\varphi)^{3/2}} \tag{5.14}$$

$$C=\frac{\sigma}{(1+\sigma^2)}\frac{\cos^2\varphi}{2m(1+\alpha)^2} \tag{5.15}$$

$$D = -\frac{1}{2}\frac{(1+\alpha)^{1/2}}{(\alpha+\sin^2\varphi)^{1/2}}. \tag{5.16}$$

(we define φ so that $m^z = +m\cos\varphi$).

The coefficient D is a diffusion coefficient in the ζ direction. In the case $\sigma \to 0$, C goes to zero but not D. This fact shows the character of the diffusion to be essentially geometric in the ζ direction.

The equation (5.12) is not integrable, but can be written in a bilinear form close to that of Hirota's theory [10, 11]; we obtain thus a nearly one-dimensional travelling wave solution for it. In laboratory coordinates it reads

$$f(x, y, t) = \frac{f_0 \,\mathrm{e}^{\eta}}{1+\mathrm{e}^{\eta}} \tag{5.17}$$

$$\eta = -A(x+qy-vt) \tag{5.18}$$

where A has the expression (4.11) and v is given by

$$v = c\left[\left(\frac{\alpha+\sin^2\varphi}{\alpha+1}\right)^{1/2} - \frac{\mu_0\delta h\cos^2\varphi}{4m(\alpha+\sin^2\varphi)^{1/2}(\alpha+1)^{3/2}} + \frac{1}{2}\left(\frac{\alpha+1}{\alpha+\sin^2\varphi}\right)q^2\right] \tag{5.19}$$

and q is an arbitrary constant. If $q \neq 0$, (5.17) is an 'oblique' nearly one-dimensional travelling wave that does not decrease in the direction $x/y = -q$ and moves at certain angle with relation to the x-axis. A solution corresponding to N such travelling waves moving all in the same direction can be constructed in the same way as in the $(1+1)$-dimensional case, and has very similar expression. It shows also the coalescence of the waves.

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Appendix. Perturbation theory in the (2 + 1)-dimensional case

We reduce here the equations (2.5) and (2.6) to the $(2+1)$ dimensions Burger equation with the conditions that $\boldsymbol{M}_n, \boldsymbol{H}_n$, $n = 0, 1, 2, 3, \ldots$ and all their ξ derivatives go to zero for $\xi \to -\infty$ except $\boldsymbol{M}_0, \boldsymbol{H}_0$ going to $\boldsymbol{m}, \boldsymbol{p}$ for $\xi \to -\infty$ and $\boldsymbol{H}_1$ going to $\boldsymbol{l}$ parallel to $\boldsymbol{m}$ as $\xi \to -\infty$.

At order zero equation (2.5) gives

$$\boldsymbol{M}_0 \wedge \boldsymbol{H}_0 - \frac{\sigma}{|\boldsymbol{M}_0|}\boldsymbol{M}_0 \wedge (\boldsymbol{M}_0 \wedge \boldsymbol{H}_0) = 0 \tag{A.1}$$

and this leads to $\boldsymbol{H}_0 = \lambda(\xi, \zeta, \tau)\boldsymbol{M}_0$. The equation (2.6) gives the system

$$M_0^x = \frac{(1+\alpha)}{(1+\lambda)}\, m^x \tag{A.2a}$$

$$M_0^y = \frac{(1+\alpha\gamma)}{(1+\lambda\gamma)}\, m^y \tag{A.2b}$$

$$M_0^z = \frac{1+\alpha\gamma}{1+\lambda\gamma} m^z \tag{A.2c}$$

where γ and α are the same as in the (1+1)-dimensional case (changing β to v). At order ε^2 we obtain from (2.5)

$$v\partial_\xi \boldsymbol{M}_0 = \boldsymbol{M}_0 \wedge (\boldsymbol{H}_1 - \lambda \boldsymbol{M}_1) - \frac{\sigma}{|\boldsymbol{M}_0|} \boldsymbol{M}_0 \wedge [\boldsymbol{M}_0 \wedge (\boldsymbol{H}_1 - \lambda \boldsymbol{M}_1)] \tag{A.3a}$$

where $\partial_\xi \boldsymbol{M}_0 = \partial \boldsymbol{M}_0 / \partial_\xi$. Multiplying by $\boldsymbol{M}_0$ (dot product) we show that $\boldsymbol{M}_0 = \text{constant} = \boldsymbol{m}$. The equation $|\boldsymbol{M}_0|^2 = |\boldsymbol{m}|^2$ is

$$\frac{(1+\alpha)^2}{(1+\lambda)^2}(m^x)^2 + \frac{(1+\alpha\gamma)^2}{(1+\lambda\gamma)^2}(m^y)^2 + \frac{(1+\alpha\gamma)^2}{(1+\lambda\gamma)^2}(m^z)^2 = (m^x)^2 + (m^y)^2 + (m^z)^2$$

consequently λ is the solution of a second-degree equation with constant coefficient. Thus $\lambda = \text{constant} = \alpha$, and $\boldsymbol{H}_0, \boldsymbol{M}_0$ are constant vectors characterizing the initial static state $\boldsymbol{M}_0 = \boldsymbol{m}\boldsymbol{H}_0 = \alpha \boldsymbol{m}$. Using this fact in (A.3a) we have that $\boldsymbol{H}_1 - \alpha \boldsymbol{M}_1$ is proportional to $\boldsymbol{m}$ and we show equation (5.8) where we introduced for convenience the factor $(1+\alpha)\mu$. Equation (2.6) gives at this order (using (5.8))

$$H_1^x = \mu n^x (f + \alpha f_0) \qquad M_1^x = \mu m^x (f_0 - f) \tag{A.3b}$$

$$H_1^y = (1+\alpha) m^y (f + \alpha\gamma f_0) \qquad M_1^y = \gamma(1+\alpha) m^y (f_0 - f) \tag{A.3c}$$

$$H_1^z = (1+\alpha) m^z (f + \alpha\gamma f_0) \qquad M_1^z = \gamma(1+\alpha) m^y (f_0 - f) \tag{A.3d}$$

with

$$f_0 = \frac{l}{m\mu(1+\alpha)} \qquad l = |\boldsymbol{l}|. \tag{A.3e}$$

At order ε^3 we obtain

$$\boldsymbol{m} \wedge (\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2) - \frac{\sigma}{m} \boldsymbol{m} \wedge (\boldsymbol{m} \wedge (\boldsymbol{H}_2 - \alpha \boldsymbol{H}_2)) = 0. \tag{A.4}$$

Thus

$$\boldsymbol{H}_2 - \alpha \boldsymbol{M}_2 = (1+\alpha)\mu g(\xi, \zeta, \tau) \boldsymbol{m} \tag{A.5}$$

with g an arbitrary function of ξ, ζ, τ. The equation (2.6) gives

$$H_2^x = \mu m^x g - \frac{\alpha}{v^2} m^y \int_{-\infty}^{\xi} f_\zeta(\xi', \zeta, \tau)\, \mathrm{d}\xi' \tag{A.6a}$$

$$H_2^y = (1+\alpha) m^y g - \frac{\alpha}{v^2} m^x \int_{-\infty}^{\xi} f_\zeta(\xi', \zeta, \tau)\, \mathrm{d}\xi' \tag{A.6b}$$

$$H_2^z = (1+\alpha) m^z g \tag{A.6c}$$

$$M_2^x = -\mu m^x g - \frac{m^y}{v^2} \int_{-\infty}^{\xi} f_\zeta(\xi', \zeta, \tau)\, \mathrm{d}\xi' \tag{A.6d}$$

$$M_2^y = -\gamma(1+\alpha) m^y g - \frac{m^x}{v^2} \int_{-\infty}^{\xi} f_\zeta(\xi', \zeta, \tau)\, \mathrm{d}\xi' \tag{A.6e}$$

$$M_2^z = -\gamma(1+\alpha) m^z g. \tag{A.6f}$$

Equation (2.5) at order ε^4 give

$$v\partial_\xi \boldsymbol{M}_1=\boldsymbol{m}\wedge(\boldsymbol{H}_3-\alpha\boldsymbol{M}_3)+\boldsymbol{M}_1\wedge\boldsymbol{H}_1-\frac{\sigma}{m}\boldsymbol{m}\wedge[\boldsymbol{m}\wedge(\boldsymbol{H}_3-\alpha\boldsymbol{H}_3)+\boldsymbol{M}_1\wedge\boldsymbol{H}_1]$$
$$+\frac{\sigma}{m}\boldsymbol{M}_1\wedge[\boldsymbol{m}\wedge(\boldsymbol{H}_1-\alpha\boldsymbol{M}_1)]. \tag{A.7}$$

Using (5.8) we show that the last term is zero and that $\boldsymbol{m}$ and $\partial_\xi \boldsymbol{M}_1$ are orthogonals

$$\boldsymbol{m}\cdot\partial_\xi\boldsymbol{M}_1=0. \tag{A.8}$$

Equation (A.8) determines v as in equation (5.9).

The remaining equations at this order give

$$M_3^x=-H_3^x-\frac{(1+\alpha)}{\alpha}m^y\mathbb{G}+m^x\mathbb{K} \tag{A.9a}$$

$$M_3^y=-\gamma H_3^y-\frac{\mu m^x}{\alpha}\mathbb{G}+\frac{\alpha m^y}{(1+\alpha)v^2}\mathbb{K}+\frac{2(1+\alpha)m^y}{v^3}\int_{-\infty}^{\xi}f_\tau(\xi',\zeta,\tau)\,\mathrm{d}\xi' \tag{A.9b}$$

$$M_3^z=-\gamma H_3^z+m^z\mathbb{K}+\frac{2(1+\alpha)m^z}{v^3}\int_{-\infty}^{\xi}f_\tau(\xi',\zeta,\tau)\,\mathrm{d}\xi' \tag{A.9c}$$

where $\mathbb{G}$ and $\mathbb{K}$ are given by

$$\mathbb{G}=\frac{\alpha}{v^2}\int_{-\infty}^{\xi}g_\zeta(\xi',\zeta,\tau)\,\mathrm{d}\xi' \tag{A.10a}$$

$$\mathbb{K}=\frac{1+\alpha}{v^2}\int_{-\infty}^{\xi}\mathrm{d}\xi'\int_{-\infty}^{\xi'}f_{\zeta\zeta}(\xi'',\zeta,\tau)\,\mathrm{d}\xi''. \tag{A.10b}$$

With these expressions and using (A.6*a*, *b*, *c*) we can show (through a laborious algebraic step) that the equation (A.7) gives

$$v\mu m^x f_\xi=-\mu m^y H_3^z+\mu m^z H_3^y+\mu m^x m^z\mathbb{G}+m^z\mathbb{F}$$
$$+\frac{\sigma}{m}\{-(1+\alpha)(m^t)^2H_3^x+\mu m^x(m^yH_3^y+m^zH_3^z)$$
$$+m^y[\mu(m^x)^2-(1+\alpha)(m^t)^2]\mathbb{G}-(m^t)^2\Phi+m^xm^y\mathbb{F}\} \tag{A.11a}$$

$$v\gamma(1+\alpha)m^yf_\xi=-m^z(1+\alpha)H_3^x+\mu m^xH_3^z-(1+\alpha)m^ym^z\mathbb{G}-m^z\Phi$$
$$+\frac{\sigma}{m}\{\mu m^ym^zH_3^z+(1+\alpha)m^xm^yH_3^x-\mu(m^l)^2H_3^y$$
$$-m^x[\mu(m^l)^2-(1+\alpha)(m^y)^2]\mathbb{G}-(m^l)^2\mathbb{F}+m^xm^y\Phi\} \tag{A.11b}$$

$$v\gamma(1+\alpha)m^zf_\xi=-\mu m^xH_3^y+(1+\alpha)m^yH_3^x-(\mu(m^x)^2-(1+\alpha)(m^y)^2)\mathbb{G}$$
$$+m^y\Phi-m^x\mathbb{F}+\frac{\sigma}{m}\{m^xm^z(1+\alpha)H_3^x-\mu[(m^x)^2+(m^y)^2]H_3^z$$
$$+\mu m^ym^zH_3^y+m^xm^ym^z(1+\alpha+\mu)\mathbb{G}+m^xm^z\Phi+m^ym^z\mathbb{F}\} \tag{A.11c}$$

where

$$(m^t)^2=(m^y)^2+(m^z)^2 \tag{A.12a}$$

$$(m^l)^2=(m^x)^2+(m^z)^2 \tag{A.12b}$$

$$\mathbb{F}=\frac{\mu\alpha m^y}{1+\alpha}\mathbb{K} \tag{A.12c}$$

$$\Phi=\frac{(1+\alpha)\mu m^x}{v^2}f(f-f_0)+\frac{2\alpha(1+\alpha)m^x}{v^3}\int_{-\infty}^{\xi}f_\tau(\xi',\zeta,\tau)\,\mathrm{d}\xi'. \tag{A.12d}$$

The next order gives the equations

$$v\partial_\xi \boldsymbol{M}_2=\boldsymbol{m}\times(\boldsymbol{H}_4-\alpha\boldsymbol{M}_4)+\boldsymbol{M}_1\times\boldsymbol{H}_2+\boldsymbol{M}_2\times\boldsymbol{H}_1$$
$$-\frac{\sigma}{m}\boldsymbol{m}\times[\boldsymbol{m}\times(\boldsymbol{H}_4-\alpha\boldsymbol{M}_4)+\boldsymbol{M}_1\times\boldsymbol{H}_2+\boldsymbol{M}_2\times\boldsymbol{H}_1]. \tag{A.13}$$

Using (5.8) and (A.5) we show that

$$\boldsymbol{M}_1\times\boldsymbol{H}_2+\boldsymbol{M}_2\times\boldsymbol{H}_1=(1+\alpha)\mu[g\boldsymbol{M}_1+f\boldsymbol{M}_2]\times\boldsymbol{m} \tag{A.14}$$

thus

$$\boldsymbol{m}\cdot\frac{\partial}{\partial_\xi}\boldsymbol{M}_2=0. \tag{A.15}$$

Now equation (A.15) yields the following condition:

$$\frac{m^x m^y}{v^2}\frac{\partial}{\partial_\zeta}f(\xi,\zeta,\tau)=0. \tag{A.16}$$

If we want non-trivial solutions we must choose $m^y=0$ and this gives equation (5.11). Finally multiplying equation (2.5) by $\boldsymbol{m}$ and isolating the order ε^6 we obtain the equation

$$(-v\partial_\xi\boldsymbol{M}_3+\partial_\tau\boldsymbol{M}_1)\cdot\boldsymbol{m}=-\boldsymbol{m}\cdot(\boldsymbol{M}_1\times\boldsymbol{H}_3+\boldsymbol{M}_3\times\boldsymbol{H}_1)$$
$$+\frac{\sigma}{m}\boldsymbol{m}\cdot[\boldsymbol{M}_1\times(\boldsymbol{m}\times(\boldsymbol{H}_3-\alpha\boldsymbol{M}_3-\mu(1+\alpha)f\boldsymbol{M}_1))]. \tag{A.17}$$

The first term in the right-hand side (using A.11*a*, *b*, *c*) is

$$\boldsymbol{m}\cdot(\boldsymbol{M}_1\times\boldsymbol{H}_3+\boldsymbol{M}_3\times\boldsymbol{H}_1)=\frac{(m^x)^2(m^t)^2}{m^2v(1+\sigma^2)}(1-\gamma)(f_0-f)\partial_\xi f \tag{A.18}$$

the second is

$$\frac{\sigma}{m}\boldsymbol{m}\cdot[\boldsymbol{M}_1\times(\boldsymbol{m}\times(\boldsymbol{H}_3-\alpha\boldsymbol{M}_3-\mu(1+\alpha)f\boldsymbol{M}_1))] \tag{A.19}$$
$$=\gamma\frac{(1+\alpha)(m^t)^2}{v}\frac{\sigma^2}{1+\sigma^2}(f_0-f)\partial_\xi f$$

and the term in the left-hand-side is

$$(-v\partial_\xi\boldsymbol{M}_3+\partial_\tau\boldsymbol{M}_1)\cdot\boldsymbol{m}=-v\left\{-\frac{\gamma m^t}{\mu m^x}\left[m^t\Phi_\xi+\frac{m}{\sigma}\mathbb{P}_\xi\right]+m^2\mathbb{K}_\xi+\frac{2(1+\alpha)(m^t)^2}{v^3}f_\tau\right\} \tag{A.20}$$

where Φ is given by (A.12*d*) and $\mathbb{P}$ by

$$\mathbb{P} = \frac{v\sigma^2}{1+\sigma^2} \frac{m^x m^t}{m^2} (1-\gamma) f_\xi \tag{A.21}$$

with (A.18), (A.19) and (A.20), we obtain the nonlinear evolution of $f(\xi, \zeta, \tau)$ (5.12).

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